### Task 1. Minimization of the functions

|  |  |  |
| --- | --- | --- |
| **variant** | **Question 1** | **Question 2** |
|  | Use the stationary conditionfor the analysis of the given function ***f****.*Check the properties of the stationary points. | Chose the function withthe given property. Note that it can be impossible. Explain this result. |
| 1 |  | The stationary condition has a unique solution, which is not a point of minimum. |
| 2 |  | The stationary condition does not have any solutions. |
| 3 |  | The stationary condition has two solutions, one of them is the point of minimum. |
| 4 |  | The stationary condition is the necessary andsufficient of minimum. |
| 5 |  | The stationary condition has an infinite set of solutions. |
| 6 |  | The stationary condition does not have any solutions butthe minimum of the function exists. |
| 7 |  | The stationary condition has three solutions: local minimum, local maximum and absolute minimum. |
| 8 |  | The stationary condition has two solutions; it is the sufficient condition of minimum. |
| 9 |  | The stationary condition for the function with two points of the absolute minimum. |
| 10 |  | The stationary condition has three solutions: local minimum, local maximum and absolute maximum. |
| 11 |  | The stationary condition has two solutions; one of them is the point of maximum. |
| 12 |  | The stationary condition does not have any solutions, butthe maximum of the function exists. |
| 13 |  | The stationary condition has three solutions: local maximum, local minimum and absolute minimum. |
| 14 |  | The stationary condition has two solutions; one of them is the point of the minimum. |
| 15 |  | The stationary condition has three solutions: local minimum, absolute maximum and absolute minimum. |
| 16 |  | The stationary condition for a function with two point of the maximum. |

### Task 2. Euler equation for Lagrange problem

Find the function  that minimizes the functional



and satisfies the boundary conditions



The values of the parameters.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **variant** |  |  |  |  |  |
| 1 |  | 0 |  | 0 | 1 |
| 2 |  | 0 | 1 | 0 | 1 |
| 3 |  | 0 |  | 0 | -1 |
| 4 |  | 0 | 1 | 0 | 1 |
| 5 |  | 0 |  | 0 | 1 |
| 6 |  | 0 | 1 | 0 | 1 |
| 7 |  | 0 | 1 | 0 | 1 |
| 8 |  | 0 |  | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 |
| 10 |  | 0 | *π/2* | 0 | 1 |
| 11 |  | 0 |  | 0 | 1 |
| 12 |  | 0 |  | 0 | 1 |
| 13 |  | 0 |  | 0 | -1 |
| 14 |  | 0 |  | 0 | -1 |
| 15 |  | 0 |  | 0 | 1 |
| 16 |  | 0 |  | 0 | 1 |
| 17 |  | 0 |  | 1 | 0 |
| 18 |  | 0 |  | 1 | 0 |
| 19 |  | 0 |  | 0 | -1 |
| 20 |  | 0 |  | 0 | -1 |
| 21 |  | 0 |  | 0 | 1 |
| 22 |  | 0 | 1 | 0 | 1 |
| 23 |  | 0 |  | 0 | -1 |
| 24 |  | 0 | 1 | 0 | 1 |
| 25 |  | 0 |  | 0 | 1 |
| 26 |  | 0 | 1 | 0 | 1 |
| 27 |  | 0 | 1 | 0 | 1 |
| 28 |  | 0 |  | 1 | 0 |
| 29 |  | 0 | 1 | 0 | 1 |
| 30 |  | 0 | π | 0 | 1 |
| 31 |  | 0 |  | 0 | 1 |
| 32 |  | 0 |  | 0 | 1 |
| 33 |  | 0 |  | 0 | -1 |

It is necessary to make the following actions:

1. Give the problem statement.
2. Determine the Euler equation.
3. Find the general solution of the Euler equation that depends from two arbitrary constants.
4. Find these constants by means of the given boundary conditions.
5. Find the corresponding solution of the boundary problem.

**Remark**. The general solution of the differential equation  is

.

The general solution of the differential equation  is



### Task 3. Special forms of the Lagrange problem

We have two Lagrange problems with the functionals

  

where the functions *F* and *G* are given (see the Table). We have also the boundary conditions



**Table of the values of the parameters.**

|  |  |  |
| --- | --- | --- |
| variant |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |
| 7 |  |  |
| 8 |  |  |
| 9 |  |  |
| 10 |  |  |
| 11 |  |  |
| 12 |  |  |
| 13 |  |  |
| 14 |  |  |
| 15 |  |  |
| 16 |  |  |

The first problem is solved with using of the Euler equation. The second problem is solved with using of the first integral. For both problems it is necessary to choose the parameters  such that the corresponding Euler equation has a solution. Find this solution.

### Task 4. Lagrange problem with two unknown functions

Find the functions   that minimize the integral



with the boundary conditions



The values of the parameters

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |  |  |
| 1 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 2 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 3 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 4 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 |  | 0 | 2 | 0 | 2 | 0 | 1 |
| 6 |  | 0 | 1 | 0 | 1 | 1 | 0 |
| 7 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 |  | 0 | 1 | 1 | 0 | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 10 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 11 |  | -1 | 0 | 0 | 1 | 0 | 1 |
| 12 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 13 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 14 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 15 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 16 |  | 0 | 1 | 0 | 1 | 1 | 0 |
| 17 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 18 |  | 0 | 1 | 0 | 2 | 0 | 1 |
| 19 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 20 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 21 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 22 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 23 |  | 0 | 2 | 0 | 2 | 0 | 1 |
| 24 |  | 0 | 1 | 0 | 1 | 1 | 0 |
| 25 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 26 |  | 0 | 1 | 1 | 0 | 1 | 0 |
| 27 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 28 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 29 |  | -1 | 0 | 0 | 1 | 0 | 1 |
| 30 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 30 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 31 |  | 0 | 2 | 0 | 1 | 0 | 1 |
| 32 |  | 0 | 1 | 0 | 1 | 0 | 1 |

Steps of the task.

1. Give the problem statement.
2. Determine the system of the Euler equations.
3. Find the general solution of this system.
4. Find the solution of the Euler equations that satisfies the boundary conditions.
5. Show the graphs of these solutions.

### Task 5. Minimization of the functional that depends from the second derivative of the unknown function

Find the function  that minimize the integral



with boundary conditions



The values of parameters.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Variant |  |  |  |  |  |  |  |
| 1 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 3 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 4 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 |  | 0 | 2 | 0 | -1 | 0 | 1 |
| 6 |  | -π | 0 | 0 | 1 | 1 | 0 |
| 7 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 |  | -1 | 1 | 1 | 0 | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 10 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 11 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 12 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 13 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 14 |  | 0 | 1 | 0 | -1 | 0 | 1 |
| 15 |  | 0 | π | 0 | 1 | 1 | 0 |
| 16 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 17 |  | -1 | 1 | 1 | 0 | 1 | 0 |
| 18 |  | 0 | 1 | 0 | 1 | 0 | 1 |

Steps of the task.

1. Give the problem statement.
2. Determine the system of the Euler – Poisson equation.
3. Find the general solution of this equation.
4. Find the solution of the Euler – Poisson equation that satisfies given boundary conditions.
5. Show the graph of this solution.

### Ostrogradsky equation for the three dimensional case and too unknown functions

We consider the functional



where Ω is three-dimensional set, *F* is given function,  Boundary conditions are given too. Write the corresponding Ostrogradsky equations



**The values of the function F:**

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 
16. 
17. 
18. 
19. 
20. 
21. 
22. 
23. 
24. 
25. 
26. 
27. 
28. 
29. 
30. 
31. 
32. 
33. 

### Task. Minimization of functionals without boundary conditions

**Variants 1-11**. Find the function, which satisfies the boundary condition  and minimize the integral

.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |
| 1 |  | *v* | 0 | 1 | 0 |
| 2 |  | -*v* | -1 | 0 | 1 |
| 3 |  | -2*v* | 0 | 1 | 0 |
| 4 |  | 2*v* | -1 | 0 | 1 |
| 5 |  | -3*v* | 1 | 0 | 0 |
| 6 |  | 3*v* | 0 | 1 | 1 |
| 7 |  | -*v* | 0 | 1 | 0 |
| 8 |  | *v* | 0 | 1 | 0 |
| 9 |  | -*v* | -1 | 0 | 1 |
| 10 |  | -2*v* | 0 | 1 | 0 |
| 11 |  | 2*v* | -1 | 0 | 1 |

**Variants 12-222**. Find the function, which satisfies the boundary condition  and minimize the integral

.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| variant |  |  |  |  |  |
| 12 |  | *v* | 0 | 1 | 0 |
| 13 |  | -*v* | -1 | 0 | 1 |
| 14 |  | -2*v* | 0 | 1 | 0 |
| 15 |  | 2*v* | -1 | 0 | 1 |
| 16 |  | -3*v* | 1 | 0 | 0 |
| 17 |  | 3*v* | 0 | 1 | 1 |
| 18 |  | -*v* | 0 | 1 | 0 |
| 19 |  | v | 0 | 1 | 0 |
| 20 |  | -v | -1 | 0 | 1 |
| 21 |  | -2v | 0 | 1 | 0 |
| 22 |  | 2v | -1 | 0 | 1 |

**Variants 23-33**. Find the function  which minimize the integral



|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant |  |  |  |  |
| 23 |  |  | 0 | 1 |
| 24 |  |  | -1 | 0 |
| 25 |  |  | 0 | 1 |
| 26 |  |  | 0 | 1 |
| 27 |  |  | 1 | 0 |
| 28 |  |  | 0 | 1 |
| 29 |  |  | 0 | 1 |
| 30 |  |  | 0 | 1 |
| 31 |  |  | -1 | 0 |
| 32 |  |  | 0 | 1 |
| 33 |  |  | 0 | 1 |

Steps of the task.

1. Determine Euler equation, and find its general solution.
2. Find the solution of this equation, which satisfies the following boundary conditions: two transversality conditions for the variants 23-33 or one transversality condition and the given boundary condition for other variants.
3. Show the graph of this solution.

**Task. Variational problem with isoperimetric condition**

Consider the problem of the minimization of the integral



with boundary conditions

  (\*)

or

  (\*\*)

and isoperimetric condition



The values of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary conditions | *A* | *b* | *M* |
| 1 | \* | 0 | π | 2 |
| 2 | \*\* | 0 | π | 1 |
| 3 | \* | π | 2π | 1 |
| 4 | \*\* | 0 | π | 2 |
| 5 | \* | 0 | π | 2 |
| 6 | \*\* | 0 | π | 3 |
| 7 | \* | 0 | π | 3 |
| 8 | \*\* | π | 2π | 1 |
| 9 | \* | π | 2π | 3 |
| 10 | \*\* | -π | 0 | 1 |
| 11 | \* | -π | 0 | 1 |
| 12 | \*\* | -π | 0 | 1/2 |
| 13 | \* | -π | 0 | 2 |
| 14 | \*\* | 0 | π | 1 |
| 15 | \* | -π | 0 | 3/2 |
| 16 | \*\* | 0 | π | 2 |
| 17 | \* | -π | 0 | 1/3 |
| 18 | \*\* | 0 | π | 3 |
| 19 | \* | 0 | π | ½ |
| 20 | \*\* | 0 | 1 | 2 |
| 21 | \* | 0 | 1 | 1 |
| 22 | \*\* | 0 | 1 | 1/2 |
| 23 | \* | 0 | 1 | 1/2 |
| 24 | \*\* | 0 | 1 | 3/2 |
| 25 | \* | 0 | 1 | 1/3 |
| 26 | \*\* | 0 | 1 | 1/2 |
| 27 | \* | 0 | 1 | 1/4 |
| 28 | \*\* | -1 | 0 | 1 |
| 29 | \* | -1 | 0 | 3 |
| 30 | \*\* | -1 | 0 | 2 |
| 31 | \* | -1 | 0 | 2 |
| 32 | \*\* | 0 | 1 | 1/3 |
| 33 | \* | -1 | 0 | 3/2 |

Steps of the task:

1. Give the concrete problem statement.
2. Write the Euler equation.
3. Verify the sign of the Lagrange multiplier with using multiplication of the Euler equation by unknown function and integration.
4. Find the general solution of the Euler equation; it depends from two constants and the Lagrange multiplier.
5. Using given boundary conditions and isoperimetric condition find three unknown constants.
6. Find the set of the solutions of the extremum conditions.
7. Calculate the value of the given integral for all solution of the conditions of the extremum.
8. Chose the minimum of these values; find the solution of the problem.

### Task. Variational problem with pointwise condition

Find the functions   that minimize the integral



on the set of the functions with boundary conditions



and additional condition

 . (\*)

The values of the parameters.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| variant | *L* | *X* | *Y* | *α* | *a*  | *b* |
| 1 | 1 | 1 | 1 | -π2 | π | -π |
| 2 | π | 1 | 2 | -1 | -1 | 1/2 |
| 3 | π/2 | 1 | 1 | -2π2 | π | -π |
| 4 | π/2 | 2 | 2 | -2 | -2 | 2 |
| 5 | 4 | 2 | 2 | -4π2 | -π | π |
| 6 | π | 1 | 1 | -1 | 2 | -2 |
| 7 | 2 | 2 | 1 | -π2 | π/2 | -π |
| 8 | π | 2 | 2 | -1 | -1 | 1 |
| 9 | 4 | π | π | -π2 | -π | π/2 |
| 10 | 1 | 1 | 1 | -1 | 1 | -1 |
| 11 | π | 1 | 1 | -π2 | π | -π |
| 12 | 2 | 1 | 1 | -1 | -1 | 1 |
| 13 | π | 1 | 2 | -π2 | -π | π/2 |
| 14 | π | 2 | 1 | -1/2 | 1/2 | -1 |
| 15 | π | 2 | 2 | -2 | -2 | 2 |
| 16 | 2 | 2 | 2 | -π2 | π/2 | -π |
| 17 | π | 2 | 1 | -1 | π/2 | -π |
| 18 | π | 1 | 2 | -π2 | 2π | -π |
| 19 | 1 | 1 | 1 | -π2 | π | -π |
| 20 | π | 1 | 2 | -1 | -1 | 1/2 |
| 21 | π/2 | 1 | 1 | -2π2 | π | -π |
| 22 | π/2 | 2 | 2 | -2 | -2 | 2 |
| 23 | 4 | 2 | 2 | -4π2 | -π | π |
| 24 | π | 1 | 1 | -1 | 2 | -2 |
| 25 | 2 | 2 | 1 | -π2 | π/2 | -π |
| 26 | π | 2 | 2 | -1 | -1 | 1 |
| 27 | 4 | π | π | -π2 | -π | π/2 |
| 28 | 1 | 1 | 1 | -1 | 1 | -1 |
| 29 | π | 1 | 1 | -π2 | π | -π |
| 30 | 2 | 1 | 1 | -1 | -1 | 1 |
| 31 | π | 1 | 2 | -π2 | -π | π/2 |
| 32 | π | 2 | 1 | -1/2 | 1/2 | -1 |
| 33 | π | 2 | 2 | -2 | -2 | 2 |

Steps of the task:

1. Denote the concrete problem statement.
2. Denote the system of the extremum conditions (concrete Euler equations with boundary and addition conditions).
3. Multiply the first Euler equation by the given parameter *a*, and second equation by *b*. Add these equalities with using of the condition (\*). Find the value *λ*.
4. Put *λ* to Euler equations.
5. Find the general solutions of two Euler equations. It equals the sum of the general solution of the corresponding homogeneous equation and the constant, which satisfies the given equation.
6. Find four constants from general solutions of Euler equations with using of the boundary conditions.
7. Put these constants to the formulas of the general solutions. It will be the solution of the problem.
8. Check the feasibility of the given constraint for it.

### Task. Optimization control problem for a system of differential equations

Consider the control system described by the differential equations



with initial conditions



The set of the admissible control is described by the inequalities



We have the problem of the minimization of the value

.

Table of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variants |  |  |  |  |
| 1 |  |  |  |  |
| 2 |  |  |  |  |
| 3 |  |  |  |  |
| 4 |  |  |  |  |
| 5 |  |  |  |  |
| 6 |  |  |  |  |
| 7 |  |  |  |  |
| 8 |  |  |  |  |
| 9 |  |  |  |  |
| 10 |  |  |  |  |
| 11 |  |  |  |  |
| 12 |  |  |  | - |
| 13 |  |  |  |  |
| 14 |  |  |  |  |
| 15 |  |  |  |  |
| 16 |  |  |  |  |
| 17 |  |  |  |  |
| 18 |  |  |  |  |

Steps of the task.

1. Write the concrete problem statement.
2. Determine the function *Н.*
3. Determine the adjoint system.
4. Determine the maximum principle.
5. Find the control from the maximum principle.
6. Write the iterative method for solving the conditions of the optimality.

### Task. Optimization control problem for a systems with fixed final time

Consider the control system described by the differential equations



with initial conditions



and final condition

  (\*)

or

  (\*\*)

The set of the admissible control is described by the inequalities



We have the problem of the minimization of the value

.

Table of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variants |  |  |  | final condition |
| 1 |  |  |  | (\*) |
| 2 |  |  |  | (\*\*) |
| 3 |  |  |  | (\*) |
| 4 |  |  |  | (\*\*) |
| 5 |  |  |  | (\*) |
| 6 |  |  |  | (\*\*) |
| 7 |  |  |  | (\*) |
| 8 |  |  |  | (\*\*) |
| 9 |  |  |  | (\*) |
| 10 |  |  |  | (\*\*) |
| 11 |  |  |  | (\*) |
| 12 |  |  |  | (\*\*) |
| 13 |  |  |  | (\*) |
| 14 |  |  |  | (\*\*) |
| 15 |  |  |  | (\*) |
| 16 |  |  |  | (\*\*) |
| 17 |  |  |  | (\*) |
| 18 |  |  |  | (\*\*) |

Steps of the task.

1. Write the concrete problem statement.
2. Determine the function *Н.*
3. Determine the adjoint system.
4. Determine the maximum principle.
5. Find the control from the maximum principle.
6. Write the iterative method for solving the conditions of the optimality.

**Task. Gradient methods**

Minimize the functional  on the given set *U*.

**Values of parameters:**

|  |  |  |
| --- | --- | --- |
| variant |  | *U* |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |
| 7 |  |  |
| 8 |  |  |
| 9 |  |  |
| 10 |  |  |
| 11 |  |  |
| 12 |  |  |
| 13 |  |  |
| 14 |  |  |
| 15 |  |  |
| 16 |  |  |
| 17 |  |  |
| 18 |  |  |
| 19 |  |  |
| 20 |  |  |
| 21 |  |  |
| 22 |  |  |
| 23 |  |  |
| 24 |  |  |
| 25 |  |  |
| 26 |  |  |
| 27 |  |  |
| 28 |  |  |
| 29 |  |  |
| 30 |  |  |
| 31 |  |  |
| 32 |  |  |
| 33 |  |  |

Steps of the task:

1. Find Gateaux derivative of the functional.
2. Determine gradient method for the problem of the minimization of the given functional without any constraints.
3. Determine gradient method for the problem of the minimization of the given functional with given constraint.